Extension theorems for analytic objects associated to foliations

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Abstract

In this paper we will establish a structure theorem concerning the extension of analytic objects associated to germs of dimension one foliations on surfaces, through one-dimensional barriers. As an application, an extension theorem for projective transverse structures is obtained.

1 Introduction

A regular one-dimensional foliation on a complex surface is given by an atlas of distinguished neighborhoods $\{U_j\}$, $j \in J$, covering the manifold, and for each $j \in J$ by a submersion $y_j \colon U_j \to \mathbb{C}$ defining the foliation, such that on each nonempty intersection $U_i \cap U_j \neq \emptyset$ we have $dy_i = g_{ij}\, dy_j$ where $g_{ij} \in \mathcal{O}^*(U_i \cap \mathrm{U}_j)$ is a not vanishing holomorphic function defined on $U_i \cap U_j$. A complex one dimensional foliation with isolated singularities on a complex surface M is a regular foliation of $M \setminus S$, where S is a discrete set of points of M. Each element of S is called an isolated *singularity* of the foliation. An elementary application of Hartog's extension theorem ([5]) shows that in the neighborhood of each singularity the foliation can be defined by a holomorphic one-form. We assume that the one-form vanishes at the singularity, otherwise the foliation would have a regular extension. Thus a foliation with a discrete set of singularities on a complex manifold M can be defined by an atlas $\{U_j\}$, $j \in J$, covering M and for each $j \in J$ a holomorphic one-form ω_j defining the foliation on U_j, such that on each nonempty intersection $U_i \cap U_j \neq \emptyset$ we have $\omega_i = g_{ij}.\omega_j$ where $g_{ij} \in \mathcal{O}^*(V_i \cap \mathrm{U}_j)$ is a not vanishing holomorphic function defined on $U_i \cap U_j \neq \emptyset$. Whenever the set S has cardinality greater than one, we say that we are dealing with a global foliation. A simple example of a global foliation is obtained by blowing-up an isolated singular point $0 \in \mathbb{C}^2$ of a foliation $\mathcal{F}$ defined in a neighborhood $0 \in U \subset \mathbb{C}^2$ by a holomorphic one-form ω, vanishing only at $0 \in \mathbb{C}^2$. Let (x, y) be coordinates of $\mathbb{C}^2$ restricted to U. Define a complex 2-manifold $\mathcal{U}$ by glueing two charts defined by the coordinates: $U_1 = (x, t)$, $U_2 = (u, y)$ such that $(x, y) \in U$ $u, t \in \mathbb{C}$, $y = t.x$, $u.t = 1$. The map $\pi_0 : \mathcal{U} \to U$ defined on these charts by $\pi_0(x, t) = (x, tx)$, $\pi_0(u, y) = (uy, y)$ is a proper holomorphic map, $D_0 = \pi_0^{-1}(0)$ is the exceptional divisor, isomorphic to an embedded projective line, and $\pi_0 : \mathcal{U} \setminus D_0 \to U \setminus \{0\}$ is a biholomorphism. On these charts $\pi_0^*(\omega) = x^\nu.\omega_1$, $\pi_0^*(\omega) = y^\nu.\omega_2$, where ν is a positive integer, depending on the algebraic multiplicity of the singularity, and ω_1, ω_2 are holomorphic 1-forms with isolated singularities. Then, the 1-forms ω_1, ω_2 satisfy $\omega_1 = g_{12}.\omega_2$, $g_{12} \in \mathcal{O}^*(U_1 \cap U_2)$ and define a foliation $\mathcal{F}_0$ on $\mathcal{U}$ called the *analytic extension* of $\pi^*\mathcal{F}$ on $\mathcal{U} \setminus D$ to $\mathcal{U}$.

We have two possibilities. Either D_0 is tangent to $\mathcal{F}_0$, *i.e.*, D_0 is a leaf plus a finite number of singularities, and in this case we say that D_0 is *nondicritical*, or D_0 is transverse to $\mathcal{F}_0$ everywhere except at a finite number of points that can be either singularities or tangency points of $\mathcal{F}_0$ with D_0. In this last case we say that D_0 is *dicritical*.

This process can be repeated at each one of the singularities, or tangency points of $\mathcal{F}_0$ with D_0. Seidenberg [10] states that by composition of a finite number of these blow-up's we can obtain a proper holomorphic map $\pi : \tilde{U} \to U$ such that $\pi^{-1}(0) = \cup_{j=0}^{m} D_j$ is a finite union of embedded projective lines with normal crossings, called the *exceptional divisor*. This map is called the *resolution morphism* of $\mathcal{F}$. Any component D_j is either invariant or everywhere transverse to the pull back foliation $\tilde{\mathcal{F}} = \pi^*(\mathcal{F})$. Any singular point of $\tilde{\mathcal{F}}$ will be *irreducible* in the following sense.

Let $\omega = a(x,y)dx + b(x,y)dy$ be a holomorphic one-form defined in a neighborhood of $0 \in \mathbb{C}^2$. We say that $0 \in \mathbb{C}^2$ is a *singular* point of ω if $a(0,0) = b(0,0) = 0$, and a *regular* point otherwise. The vector field $X = (-b(x,y), a(x,y))$ is in the kernel of ω. The nonsingular orbits of X are the leaves of the foliation.

We say that $0 \in \mathbb{C}^2$ is an *irreducible* singular point of ω if the eigenvalues λ_1, λ_2 of the linear part of X at $0 \in \mathbb{C}^2$ satisfy one of the following conditions:

(1) $\lambda_1.\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \notin \mathbb{Q}_+$

(2) either $\lambda_1 \neq 0$ and $\lambda_2 = 0$, or viceversa.

In case (1) there are two invariant curves tangent to the eigenvectors corresponding to λ_1 and λ_2. In case (2) there is an invariant curve tangent at $0 \in \mathbb{C}^2$ to the eigenspace corresponding to λ_1. These curves are called *separatrices* of the foliation.

Suppose that $0 \in \mathbb{C}^2$ is either a regular point or an irreducible singularity of a foliation $\mathcal{I}$. It is possible to show that in suitable local coordinates (x,y) in a neighborhood $0 \in U \in \mathbb{C}^2$ of the origin, we have the following local normal forms for the one-forms defining this foliation:

(Reg) $dx = 0$, whenever $0 \in \mathbb{C}^2$ is a regular point of $\mathcal{I}$.

and if $0 \in \mathbb{C}^2$ is an irreducible singularity of $\tilde{\mathcal{F}}$, then either

(Irr.1) $xdy - \lambda ydx + \eta_2(x,y) = 0$ where $\lambda \in \mathbb{C}\backslash\mathbb{Q}_+$, $\eta_2(x,y)$ is a holomorphic one-form with a zero of order ≥ 2 at $(0,0)$. This is called *nondegenerate singularity*. Such a singularity is *resonant* if $\lambda \in \mathbb{Q}_-$ and *hyperbolic* if $\lambda \notin \mathbb{R}$, or

(Irr.2) $y^{t+1}dx - [x(1+\lambda y^t) + A(x,y)]dy = 0$, where $\lambda \in \mathbb{C}$, $t \in \mathbb{N} = \{1,2,3,\dots\}$ and $A(x,y)$ is a holomorphic function with a zero of order $\geq t+2$ at $(0,0)$. This is called *saddle-node singularity*. The *strong separatrix* of the saddle-node is given by $\{y = 0\}$. If the singularity admits another separatrix then it is necessarily smooth and transverse to the strong separatrix, it can be taken as the other coordinate axis and will be called *central* manifold of the saddle-node.

In the last two cases we have $\{y = 0\} \subset \mathrm{sep}(\mathcal{I}, U) \subset \{xy = 0\}$, where $\mathrm{sep}(\mathcal{I}, U)$ denotes the union of separatrices of $\mathcal{I}$ through $0 \in \mathbb{C}^2$.

A *fundamental domain* of $\mathcal{I}|_U$ at $0 \in \mathbb{C}^2$ is a subset $\mathcal{D} \subset \mathbb{C}^2$ consisting of:

(E.0) In the regular case: a neighborhood of $0 \in \mathbb{C}^2$ minus a codimension one submanifold passing through $0 \in \mathbb{C}^2$, transverse to the foliation.

(E.1) A neighborhood of the singularity minus its separatrices in case the singularity is nondegenerate nonresonant.

(E.2) A neighborhood of the singularity minus its separatrices, union a neighborhood of an annulus around $0 \in \mathbb{C}^2$ contained in one of the separatrices, in case the singularity is resonant.

(E.3) A neighborhood of the singularity minus its separatrices, union a neighborhood of an annulus around $0 \in \mathbb{C}^2$ contained in the strong separatrix, in case the singularity is a saddle-node.

Conditions (E.2) and (E.3) are related to the fact that, in the resonant case and in the saddle-node case, the holonomy of the mentioned separatrix characterizes the analytical type of the foliation (cf. [6], [7]).

1.1 The Globalization Theorem

Consider now an arbitrary germ of an analytic foliation $\mathcal{F}$ at an isolated singularity $0 \in \mathbb{C}^2$ with resolution morphism $\pi\colon \tilde{U} \to U$. A *separatrix* of $\mathcal{F}$ at $0 \in \mathbb{C}^2$ is the germ at $0 \in \mathbb{C}^2$ of an irreducible analytic curve which is invariant by $\mathcal{F}$. It follows from the resolution theorem that for a small enough neighborhood $0 \in U \subset \mathbb{C}^2$ any separatrix of $\mathcal{F}$ can be represented in U by an irreducible analytic curve passing through $0 \in \mathbb{C}^2$ which is the closure of a leaf of $\mathcal{F}|_U$. We will write $\operatorname{sep}(\mathcal{F}, U)$ to denote the union of these separatrices. By Newton-Puiseux parametrization theorem, if U is small enough, there is an analytic injective map $f\colon \mathbb{D} \to U$ from the unit disk $\mathbb{D} \subset \mathbb{C}$ onto the separatrix, mapping the origin to $0 \in \mathbb{C}^2$, and nonsingular outside the origin $0 \in \mathbb{D}$. Therefore a separatrix locally has the topology of a punctured disk. We shall say that the separatrix is *resonant* if for any loop in the punctured disk that represents a generator of the homotopy of the leaf, the corresponding holonomy map is a resonant diffeomorphism. Choose a holomorphic vector field X which generates the foliation $\mathcal{F}|_U$, and has an isolated singularity at $0 \in \mathbb{C}^2$. Then, the separatrix is called *resonant* if the loop γ generating the homotopy of the leaf in the separatrix satisfies $\exp \int_\gamma \operatorname{tr}(DX)$ is a root of the unity.

The main concept we introduce is the following:

Definition 1. A *fundamental domain* of $\mathcal{F}|_U$ at $0 \in \mathbb{C}^2$ is a subset $\mathcal{D} \subset U$ which is :

(i) A fundamental domain of $\mathcal{F}|_U$ at $0 \in \mathbb{C}^2$ whenever $0 \in \mathbb{C}^2$ is either a regular point or an irreducible singularity, and

(ii) In case $0 \in \mathbb{C}^2$ is a not irreducible singularity, a subset $\mathcal{D} \subset \mathbb{C}^2$ written as $\mathcal{D} = (U \setminus \operatorname{sep}(\mathcal{F}, U)) \cup \mathcal{S}$, where $\mathcal{S} \subset U$ is the union of ring neighborhoods of loops γ, one for each resonant separatrix.

It is important to remark that the pull-back of a fundamental domain by the resolution morphism, is a fundamental domain for some singularities of $\tilde{\mathcal{F}}$, but not necessarily for all of them. This is the case, for instance, for saddle-nodes with strong manifold tangent to the resolution divisor.

Let U be a neighborhood of $0 \in \mathbb{C}^2$, as above. A meromorphic q-form ξ defined on $U \setminus \mathrm{sep}(\mathcal{F}, U)$ is called *extensible with respect to* $\mathcal{F}|_U$ in U if any extension of ξ to a fundamental domain of $\mathcal{F}|_U$ extends as a meromorphic q-form to U. We will say also that ξ is *extensible with respect to the germ* $\mathcal{F}$ at $0 \in \mathbb{C}^2$ if it is extensible with respect to $\mathcal{F}|_U$ in some neighborhood $0 \in U \subset \mathbb{C}^2$.

In general it is a not trivial task to prove that a q-form is extensible with respect to a local foliation. We show in section 4 that one-forms associated to projective transverse structures of a foliation $\mathcal{I}$ are extensible with respect to $\mathcal{I}$.

Let U be a neighborhood of $0 \in \mathbb{C}^2$, as above. A meromorphic q-form ξ defined on $U \setminus \mathrm{sep}(\mathcal{F}, U)$ is called *infinitesimally extensible with respect to* $\mathcal{F}|_U$ in U if $\tilde{\xi} := \pi^*\xi$ is an extensible q-form with respect to $\tilde{\mathcal{F}}|_{\pi^{-1}(U)}$ at a generic point on each dicritical component and in a neighborhood of each irreducible singular point of $\tilde{\mathcal{F}}$.

A natural question is to find extension theorems for general germs of foliations. We will show next that for any germ of a foliation it is enough to check extensibility at the irreducible singularities produced in the process of desingularization.

Theorem 1 (Globalization theorem)**.** *Let $\mathcal{F}$ be the germ of a holomorphic foliation with an isolated singularity at $0 \in \mathbb{C}^2$. For a small enough neighborhood $0 \in U \subset \mathbb{C}^2$ any meromorphic q-form infinitesimally extensible with respect to $\mathcal{F}$ in U is extensible.*

2 Resolution of singularities

2.1 The Index theorem

Let σ be a Riemann surface embedded in a two dimensional manifold S ; $\mathcal{F}$ a foliation on S which leaves σ invariant and $q \in \sigma$. There is a neighborhood of q where σ can be expressed by $(f = 0)$ and $\mathcal{F}$ is induced by the holomorphic 1-form ω written as $\omega = hdf + f\eta$. Then we can associate the following index:

$$i_q(\mathcal{F}, \sigma) := -\mathrm{Residue}_q(\frac{\eta}{h})|_\sigma$$

relative to the invariant submanifold σ. A nondegenerate singularity in the form (Irr.1) has two invariant manifolds crossing normally, they correspond to the x and y-axes. In this case if σ is locally $(y = 0)$ and $q = 0$, this index is equal to λ (quotient of eigenvalues). The saddle-node in (Irr.2) has an invariant manifold corresponding to the x-axis and, depending on the higher order terms, it may not have another invariant curve (see [7]). In the case of a saddle-node, if σ is equal to $(x = 0)$ and $q = 0$, this index is λ, and if σ is equal to $(y = 0)$ and $q = 0$, this index is zero. At a regular point q of $\mathcal{F}$ the index is zero. The index theorem of [2] asserts that the sum of all the indices at the points in σ is equal to the self-intersection number $\sigma \cdot \sigma$:

$$\sum_{q \in \sigma} i_q(\mathcal{F}, \sigma) = \sigma \cdot \sigma.$$

2.2 Resolution of singularities: linear chains

Suppose $\mathcal{F}$ is a complex one-dimensional foliation defined on an open neighborhood $0 \in U \subset \mathbb{C}^2$. The resolution process of $\mathcal{F}$ at $0 \in \mathbb{C}^2$ can be described and ordered as follows. The blow-up of $\mathcal{F}$ at $0 \in \mathbb{C}^2$ is $(U_0, \pi_0, D_0, \mathcal{F}_0)$ where $\pi_0 : U_0 \to U$ is the usual blow-up map (see § 1). Then, U_0 is a complex 2-manifold, $D_0 = \pi_0^{-1}(0) \subset U_0$ is an embedded projective line called the *exceptional divisor*, and the restriction of the map π_0 to $U_0 \setminus D_0$ is a biholomorphism from $U_0 \setminus D_0$ to $U \setminus \{0\}$. Moreover $\mathcal{F}_0$ is the analytic foliation on U_0 obtained by extension to D_0 of $(\pi_0|_{U_0 \setminus D})^*\mathcal{F}$, as defined in the Introduction. We also observe that the Chern class of the normal bundle to $D_0 \subset U_0$ is -1. We have two possibilities. Either D_0 is tangent to $\mathcal{F}_0$, i.e. D_0 is a leaf plus a finite number of singularities, and in this case we say that D_0 is *nondicritical*, or D_0 is transverse to $\mathcal{F}_0$ everywhere except at a finite number of points that can be either singularities or tangency points of $\mathcal{F}_0$ with D_0. In this last case we say that D_0 is *dicritical*.

Proceeding by induction we define the step $\underline{0}$ as the first blow-up $(U_0, \pi_0, D_0, \mathcal{F}_0)$. We assume that $(U_k, \pi_k, D_k, \mathcal{F}_k)$ has been already defined, where $\pi_k : U_k \to U$ is a holomorphic map, such that $D_k = \pi_k^{-1}(0)$ is a divisor, union of a finite number of embedded projective lines with normal crossing. The crossing points of D_k are called *corners*. The restriction of π_k to $U_k \setminus D_k$ is a biholomorphism from $U_k \setminus D_k$ to $U \setminus \{0\}$. The foliation $\mathcal{F}_k$ on U_k is the analytic extension to D_k of the foliation $(\pi_k|_{U_k \setminus D_k})^*\mathcal{F}$.

Let $p_0 : \tilde{U}_k \to U_k$ be the blow-up at a point $r \in D_k$, outside the corners. Let $P_0 = p_0^{-1}(r)$ be the exceptional divisor. We write $\tilde{D}_k = \overline{p_0^{-1}(D_k \setminus \{r\})}$ and $\tilde{r} = P_0 \cap \tilde{D}_k$. If P is the irreducible component of D_k containing r we will denote by $\tilde{P} = \overline{p_0^{-1}(P \setminus \{r\})}$. Then it is easy to see ([2]) that $i_{\tilde{r}}(\tilde{P}) = i_r(P) - 1$. Using the fact that the restriction of p_0 to $\tilde{U}_k \setminus P_0$ is a biholomorphism onto $U_k \setminus \{r\}$ we will say that r becomes $\tilde{r}$ after one blow-up and also simplify notations identifying $\tilde{D}_k$ with D_k, $\tilde{P}$ with P and $\tilde{r}$ with r. Thus in the new notation, $(\pi_k \circ p_0)^{-1}(0) = D_k \cup P_0$ and we will say that $r = P_0 \cap D_k$ was blown-up once.

We proceed to define $(U_{k+1}, \pi_{k+1}, D_{k+1}, \mathcal{F}_{k+1})$ as follows. Let $\tau_k \subset D_k$ be the set of points outside the corners of D_k, that are either tangency points of $\mathcal{F}_k$ with D_k or not irreducible singular points of $\mathcal{F}_k$. Let $r \in \tau_k$. We introduce at r a *linear chain* $\mathcal{C}(r)$ *with origin at* $r \in D_k$, by means of a sequence of blow-up's, first at the point r, the precise number of times necessary to become either irreducible, or regular and then at any reducible corner produced in this way. The resolution theorem of Seidenberg [10] guarantees that after a finite number of blow-up´s all corners obtained in this process will be either irreducible singular points or regular points.

The linear chain $\mathcal{C}(r)$ can be seen as an ordered finite sequence of embedded projective lines: $P_m > P_{m-1} > ... > P_1$ where $r = D_k \cap P_m$ and if $i > j$ and $P_i \cap P_j \neq \emptyset$ then $i = j + 1$ and $P_j \cap P_{j+1}$ is just one point. For any $l = 1, ..., m-1$ write $r_l = P_l \cap P_{l+1}$. Two invariants can be associated to $\mathcal{C}(r)$. One is the *order* n_r of $\mathcal{C}(r)$ defined as the the minimun number of times that was necessary to blow-up r in order to become irreducible, and the *length* m of the linear chain. Given any number $1 \leq t < m$ we will say that the sequence $P_t > P_{t-1} > ... > P_1$ is a linear chain $\mathcal{C}(r_t)$ of length t with origin at $r_t \in P_t \cap P_{t+1}$. We will write $|\mathcal{C}(r)| = \cup_{j=1}^m P_j$ to denote the *support* of the chain $\mathcal{C}(r)$.

Let $-k_l = P_l.P_l$ be the self intersection number of P_l in the linear chain $\mathcal{C}(r)$. The sequence of numbers $n_r.k_m....k_1$ belongs to the collection $\mathcal{A}$ of numbers defined as follows. Start with $1.1 \in \mathcal{A}$ and assume that $a_0.a_t.a_{t-1}....a_1$ belongs to $\mathcal{A}$. Then $(a_0 + 1.1.(a_t + 1).a_{t-1}....a_1 \in \mathcal{A}$, and $a_0.a_t...(a_{j+1} + 1).1.(a_j + 1)....a_1 \in \mathcal{A}$.

Lemma 1 ([2]). *Suppose that $a_0.a_t.a_{t-1}....a_1 \in \mathcal{A}$. Then*

$$a_0 = [a_t, a_{t-1}, ..., a_2, a_1] := \cfrac{1}{a_t - \cfrac{1}{a_{t-1} - \cfrac{1}{\ddots \; \cfrac{1}{a_2 - \cfrac{1}{a_1}}}}}$$

We also have the following

Lemma 2 ([2]). *If $a_0.a_t...a_1 \in \mathcal{A}$, then*
a) $[a_l, ..., a_h] > 0$ if $1 \leq h \leq l \leq t$ and $t \geq 2$
b) $0 < [a_t, ..., a_{t-i}] < [a_t, ..., a_1]$ for $0 \leq i \leq t-2$

Let $p_1, p_2, ..., p_u$ be the ordered sequence of blow-up's that created the linear chain $\mathcal{C}(r)$, then the composition $p = p_u \circ ... \circ p_2 \circ p_1$, is a map $p : \tilde{U}(r) \to U_k$ for which $p^{-1}(r) = |\mathcal{C}(r)| = \cup_{l=1}^{m} P_l$, where each P_l is an embedded projective line and $r_l = P_l \cap P_{l+1}$ is just a point, and $r_m = P_m \cap D_k$ where we are making the identification $D_k \equiv \overline{p^{-1}(D_k \setminus \{r\})}$, using the fact that the restriction of p to $\tilde{U}(r) \setminus |\mathcal{C}(r)|$ is a biholomorphism onto $U_k \setminus \{r\}$.

Repeating this process at each of the points of τ_k we obtain, by composition of these maps, a holomorphic map $p_{k+1} : U_{k+1} \to U_k$ such that $p_{k+1}^{-1}(\tau_k) = \cup_{r \in \tau_k} |\mathcal{C}(r)|$, a union of the supports of the linear chains with origin at the points in τ_k. Moreover $p_{k+1} : U_{k+1} \setminus \cup_{r \in \tau_k} |\mathcal{C}(r)| \to U_k \setminus \tau_k$ is a biholomorphism. Define $D_{k+1} := D_k \cup_{r \in \tau_k} |\mathcal{C}(r)|$ where we have identified D_k with $\overline{p_{k+1}^{-1}(D_k \setminus \tau_k)}$. Finally, we define $\pi_{k+1} : U_{k+1} \to U$ by $\pi_{k+1} := \pi_k \circ p_{k+1}$, and $\mathcal{F}_{k+1}$ as the analytic extension of $(p_{k+1}|_{U_{k+1} \setminus D_k})^* \mathcal{F}_k$ to D_{k+1}.

The theorem of Seidenberg asserts that this process ends after a finite number of steps. We observe that the dicritical components in the final configuration are disjoint, have no singularities and are everywhere transverse to the foliation. The resolution of $\mathcal{F}$ at $0 \in \mathbb{C}^2$ is $(U_n, \pi_n, D_n, \mathcal{F}_n)$ if all the singularities of $\mathcal{F}_n$ in D_n are irreducible but at least one singularity of $\mathcal{F}_{n-1}$ in D_{n-1} is not irreducible.

3 Proof of the Globalization theorem

Let U be a neighborhood of $0 \in \mathbb{C}^2$, small enough such that any separatrix of $\mathcal{F}_U$ is an irreducible curve, union of a leaf of $\mathcal{F}_U$ and the point $0 \in \mathbb{C}^2$. Let ξ be a meromorphic q-form defined on a fundamental domain $\mathcal{D} = V \setminus \text{sep}(\mathcal{F}, V) \cup \mathcal{S}$ of $\mathcal{F}|_V$ at $0 \in \mathbb{C}^2$, where $\mathcal{S}$ is the union of ring neighborhoods of generating cycles, one for each separatrix. Consider a generic linear chain created at the k-step in the process of resolution with origin at a point $r \in P \subset D_k$, $\mathcal{C}(r) = (P_l)_{l=1}^{m}$, where P is the irreducible component of D_k containing r. As before denote by $p : U(r) \to U_k$ the sequence

of blow-up's that defined $\mathcal{C}(r)$. Then $p_k \circ p : \tilde{U}(r) \to U$ defines $\tilde{\mathcal{F}}(r) = (p_k \circ p)^* \mathcal{F}|_V = p^*(p_k^*(\mathcal{F})) = p^*(\mathcal{F}_k)$ in the neighborhood $\tilde{U}(r)$ of $|\mathcal{C}(r)| \cup D_k$. Define also $\tilde{\mathcal{D}} = (p_k \circ p)^{-1}(\mathcal{D})$. We will write $P = P_{m+1}$, $\tilde{U} = \tilde{U}_r$, $\tilde{\mathcal{F}}$=$\tilde{\mathcal{F}}(r)$, $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}(r)$ and $r = r_{m+1}$, for simplicity.

Then $\tilde{\mathcal{D}} \subset \tilde{U}$ can be written as $\tilde{\mathcal{D}} = \tilde{U} \backslash \mathrm{sep}(\tilde{\mathcal{F}}, \tilde{U}) \cup \tilde{\mathcal{S}}$, where $\tilde{\mathcal{S}}$ is the union of ring neighborhoods of generating loops γ, one for each resonant separatrix not contained in $|\mathcal{C}(r)| \cup D_k$ and $\mathrm{sep}(\tilde{\mathcal{F}}, \tilde{U})$ is the union of $|\mathcal{C}(r)|$ and the separatrices of $\tilde{\mathcal{F}}$. By hypothesis the meromorphic q-form $\tilde{\xi}$, defined on $\tilde{\mathcal{D}}$ is extensible with respect to $\tilde{\mathcal{F}}$ on $\tilde{U}$ at each singularity of $\tilde{\mathcal{F}}$ and at a generic point in each dicritical component of $\mathcal{C}(r)$. However, $\tilde{\mathcal{D}}$ is not necessarily a fundamental domain for each singularity of $\tilde{\mathcal{F}}$.

Denote by $-k_l$, where k_l is a positive integer, the self-intersection number of $P_l \subset U_k$. Let $\zeta_l \subset P_l$ be the set of singular points of $\tilde{\mathcal{F}}$ in P_l which are not corners of $\mathcal{C}(r)$ and have a positive index relative to P_l. Clearly $c_l := \sum_{p \in \zeta_l} i_p(P_l)$ is a not negative number and we define $\bar{k}_l = k_l + c_l$, for $l = 1, ..., m$. Define also $\zeta = \cup_{l=1}^m \zeta_l$, and assume that any singularity in $\cup_{l=1}^m P_l$ outside ζ is irreducible.

We will say that $\tilde{\xi}$ has a meromorphic extension to a neighborhood of $|\mathcal{C}(r)| \backslash \zeta$ if for any compact subset $K \subset |\mathcal{C}(r)| \setminus \zeta$ there is a neighborhood $K \subset \tilde{U}_K \subset \tilde{U}$ in $\tilde{U}$ and a meromorphic extension of $\tilde{\xi}$ to $\tilde{U}_K$.

Definition 2 (minimal chain)**.** A linear chain $\mathcal{C}(r) = (P_l)_{l=1}^m$ is called *minimal* if any corner $r_l = P_l \cap P_{l+1}$, $l = 1, ..., m$ is of one of the following types:

(i) a regular point.

(ii) a saddle-node singularity with $i_{r_l}(P_{l+1}) = 0$

(iii) a resonant singularity with $i_{r_l}(P_l) + i_{r_{l-1}}(P_l) = -\bar{k}_l$, if $l > 1$, and $i_{r_1}(P_1) = -\bar{k}_1$ for $l = 1$.

Proposition 1. *Suppose that $\mathcal{C}(r) = (P_l)_{l=1}^m$ is a linear chain of $\tilde{\mathcal{F}}$ containing no dicritical components such that any singularity in $|\mathcal{C}(r)| \backslash \zeta$ is irreducible. Assume there is a meromorphic q-form $\tilde{\xi}$ defined on $\tilde{\mathcal{D}}$. Then, either $\tilde{\xi}$ has a meromorphic extension to a neighborhood of $|\mathcal{C}(r)| \setminus \zeta$, or $\mathcal{C}(r)$ is a minimal chain.*

Proof. We proceed by induction on the length m of the linear chain $\mathcal{C}(r)$. Suppose that $m = 1$. Then $r = P_1 \cap P_2$. Assume first that r is a nondegenerate nonresonant singularity, then as $\tilde{\xi}$ is extensible with respect to $\tilde{\mathcal{F}}$ at r it extends as a meromorphic q-form to a neighborhood of r. By Levi's extension theorem ([11]) there is an arbitrarily small neighborhood N_1 of the separatrices transverse to P_1 different than P_2 such that $\tilde{\xi}$ extends as a meromorphic q-form to $\tilde{U} \setminus N_1$.

Given an irreducible singular point $p \in P_1 \setminus \zeta_1$, then there are two possibilities. Either $i_p(\tilde{\mathcal{F}}, P_1) \neq 0$, and so there is a separatrix $\tilde{s}$ of $\tilde{\mathcal{F}}$ at p, transverse to P_1. If p is either a resonant or a saddle-node then the holonomy of $\tilde{s}$ is resonant and so $\tilde{\xi}$ is defined in a fundamental domain at p, consequently it extends as a meromorphic q-form to a neighborhood of p. Similarly, if p is not degenerate, and since the q-form $\tilde{\xi}$ is extensible with respect to $\tilde{\mathcal{F}}$, at p, then it extends as a meromorphic q-form to a neighborhood of p. On the other hand, if $i_p(\tilde{\mathcal{F}}, P_1) = 0$ then p is a saddle-node with its strong invariant manifold contained in P_1. Moreover $\tilde{\xi}$ is defined in a fundamental domain at p and since it is extensible with respect to $\tilde{\mathcal{F}}$ at p, it can be extended to a

neighborhood of p as a meromorphic one-form. Thus, we can assume that r is either a saddle-node or a resonant singularity. Suppose that $\mathcal{C}(r)$ is not minimal. Then $i_r(P_2) \neq 0$ *and* $i_r(P_2) \neq -1/\bar{k}_1$, and r is either a saddle-node singularity with index $i_r(P_1) = 0$ or a resonant singularity with $i_r(P_1) \neq -\bar{k}_1$. In both cases we have that $i_r(P_1) + c_1 \neq -k_1$, and so by the index theorem there is a singular point $p \in P_1 \setminus \{r, \zeta_1\}$ with $i_p(P_1) \neq 0$ not positive. By hypothesis p is irreducible, then there is a separatrix $\tilde{s}$ of $\tilde{\mathcal{F}}$ at p, transverse to P_1. If p is either a resonant or a saddle-node then the holonomy of $\tilde{s}$ is resonant and so $\tilde{\xi}$ is defined in a fundamental domain at p and so it extends as a meromorphic q-form to a neighborhood of p. Similarly, if p is not degenerate the q-form $\tilde{\xi}$, extensible with respect to $\tilde{\mathcal{F}}$ at p, extends as a meromorphic q-form to a neighborhood of p. Therefore, by Levi´s extension theorem ([11]) we can extend $\tilde{\xi}$ as close to r as we wish. Since r is either a saddle-node with the strong separatrix tangent to P_1 or a resonant singularity, then $\tilde{\xi}$ is already defined in a fundamental domain at r and so it can be extended to a neighborhood of r. From this we obtain that $\tilde{\xi}$ extends to a neighborhood of $P_1 \setminus \zeta_1$.

Fix any integer t, $2 \leq t \leq m$, and assume that the alternative stated in the theorem holds true for linear chains of length $t-1$. Then we have two possibilities: either a) $\tilde{\xi}$ has been extended to $P_1 \cup ... \cup P_{t-1} \setminus \zeta_1 \cup ... \cup \zeta_{t-1}$, or b) the linear chain $\mathcal{C}(r_{t-1})$ is minimal. Consider the linear chain $\mathcal{C}(r_t)$ of length t and assume that $i_{r_t}(P_{t+1}) \neq 0$. If r_t is a not degenerate, nonresonant singularity, then $\tilde{\xi}$ extends to a neighborhood of r_t and from there to a neighborhood of $P_1 \cup ... \cup P_t \setminus \zeta_1 \cup ... \cup \zeta_t$. We can then assume that r_t is either a saddle-node with $i_{r_t}(P_t) = 0$ or a resonant singularity. If case a) happens then $\tilde{\xi}$ is well defined in a neighborhood of r_{t-1}, then by Levi's theorem $\tilde{\xi}$ will extend as close to r_t as desired. Then $\tilde{\xi}$ is defined in a fundamental domain at r_t and therefore extends as a meromorphic q-form to a neighborhood of r_t and thus to $P_1 \cup ... \cup P_t \setminus \zeta_1 \cup ... \cup \zeta_t$. In case b) we have that either $i_{r_{t-1}}(P_t) = 0$, or $i_{r_{t-1}}(P_t) = -[\bar{k}_{t-1}, ..., \bar{k}_h]$, where h is the greatest positive integer $2 \leq h \leq t-1$ such that $i_{r_{h-1}}(P_h) = 0$. It is easy to see from Lemma 2, that $-[\bar{k}_{t-1}, ..., \bar{k}_h] > -\bar{k}_t$. Thus $\bar{k}_t + i_{r_{t-1}}(P_t) > 0$. Suppose further that $i_{r_t}(P_{t+1}) \neq 0$ and $i_{r_t}(P_{t+1}) \neq -1/\bar{k}_t + i_{r_{t-1}}(P_t)$, then either r_t is a saddle-node with $i_{r_t}(P_t) = 0$ or it is a resonant singularity with $i_{r_t}(P_t) \neq -\bar{k}_t - i_{r_{t-1}}(P_t)$. In any case $i_{r_t}(P_t) + i_{r_{t-1}}(P_t) \neq -\bar{k}_t$ and therefore there exists $p \in P_t \setminus \{r_{t-1}, r_t, \zeta_t\}$ such that $i_p(P_t) \neq 0$. Thus we can extend ξ through p to a neighborhood of r_{t-1} and r_t and then to a neighborhood of $P_1 \cup ... \cup P_t \setminus \zeta_1 \cup ... \cup \zeta_t$. It is clear that the only alternative left is $i_{r_t}(P_{t+1}) = 0$ or $i_{r_t}(P_{t+1}) = -1/\bar{k}_t + i_{r_{t-1}}(P_t)$. This last equation is equivalent to $i_{r_t}(P_t) + i_{r_{t-1}}(P_t) = -\bar{k}_t$.

□

Lemma 3. *Suppose that in the linear chain $\mathcal{C}(r_m)$, P_{m+1} is dicritical and the P_l, $l = 1, ..., m$ are nondicritical. Then $\tilde{\xi}$ can be extended to a neighborhood of $P_1 \cup ... \cup P_m \setminus \zeta$.*

Proof. If $\tilde{\xi}$ extends to a neighborhood of $P_1 \cup ... \cup P_{m-1} \setminus \zeta$, then in particular it is well defined in a neighborhood of r_{m-1}. Thus it can be extended to a neighborhood of r_m. Suppose on the other hand that the linear chain $\mathcal{C}(r_{m-1})$ is minimal. Then either $i_{r_{m-1}}(P_m) = 0$, or $i_{r_{m-1}}(P_m) = -[\bar{k}_{m-1}, ..., \bar{k}_h]$, where h is the greatest positive integer $2 \leq h \leq m-1$ such that $i_{r_{h-1}}(P_h) = 0$. Since $[\bar{k}_m, ..., \bar{k}_h] > 0$ we have that $\bar{k}_m > [\bar{k}_{m-1}, ..., \bar{k}_h]$, and so $i_{r_{m-1}}(P_m) > -\bar{k}_m$. This is the same as $i_{r_{m-1}}(P_m) + c_m > -k_m$. By the index theorem there is $p \in P_m \setminus \{r_{m-1}, r_m, \zeta_m\}$ such that

$i_p(P_m) \neq 0$. Thus $\tilde{\xi}$ can be extended through p to a neighborhood of r_m and r_{m-1} and from there to a neighborhood of $P_1 \cup ... \cup P_m \setminus \zeta$. □

Lemma 4. *Suppose that in the linear chain $\mathcal{C}(r_m)$, P_1 is dicritical and the P_l, $l \neq 1$, are nondicritical. Then either $\tilde{\xi}$ extends to $P_1 \cup ... \cup P_m \setminus \zeta$, or the chain $\mathcal{C}_2(r_m) = (P_l)_{l=2}^m$ is minimal.*

Proof. This is clearly a consequence of the Proposition as $i_{r_1}(P_2) = 0$, and $\tilde{\xi}$ extends to $P_1 \setminus \{r_1\}$. □

Proof of Theorem 1. Let us now prove Theorem 1. Suppose $(U_n, \pi_n, D_n, \mathcal{F}_n)$ is the resolution of $\mathcal{F}$ at $0 \in \mathbb{C}^2$. Consider a linear chain $\mathcal{C}(r) = (P_l)_{l=1}^m$ of order k with origin at a point $r \in D_{n-1}$. In this case we can take $\zeta = \emptyset$ in Proposition 1 and assume that $\mathcal{C}(r)$ is minimal. From (ii) in the definition of minimal chain we obtain that $i_r(P) = -[k_m, ..., k_h] \geq -[k_m, ..., k_1]$. Since $k = [k_m, ..., k_1]$ is the number of times that the point r was blown-up to create $\mathcal{C}(r)$, we obtain that the index of r before the creation of $\mathcal{C}(r)$ is $i_r(P)_b = i_r(P) - k \geq 0$. Thus, for the linear chains in $D_{n-1} \setminus D_{n-2}$ the origins of the linear chains in $D_n \setminus D_{n-1}$ contribute with a positive index. Thus we can apply again the Proposition 1.

Finally, we consider $\mathcal{F}_0$. Let $\mathcal{C}(q_1), ..., \mathcal{C}(q_t)$ be all linear chains starting at the reduced singularities in D_0. Let l_i denote the order of $\mathcal{C}(q_i)$. Then $i_{q_i}(D_0) \geq 0$, for $i = 1, ..., t$. Since the self-intersection number of D_0 is -1, there must exist a point $s \in D_0 \setminus \{q_1, ..., q_t\}$ such that $i_s(D_0) \neq 0$. Therefore ξ can be extended to a neighborhood of $D_0 \setminus \{q_1, ..., q_t\}$ and from there to the whole of $D_1 \cup ... \cup D_n$. □

4 Foliations with projective transverse structure

The Globalization Theorem has some important consequences in the study of transverse structure of holomorphic foliations with singularities. We focus on the case of projective transverse structures, which is the general case in codimension one (the affine and additive remaining cases are viewed as subcases).

4.1 Transversely projective foliations with singularities

Let $\mathcal{F}$ be a codimension one holomorphic foliation on a connected complex manifold M^m, of dimension $m \geq 2$, having singular set $\operatorname{sing}(\mathcal{F})$ of codimension ≥ 2. The foliation $\mathcal{F}$ is *transversely projective* if there is an open cover $\{U_j,\, j \in J\}$ of $M \setminus \operatorname{sing}(\mathcal{F})$ such that on each U_j the foliation is given by a holomorphic submersion $f_j \colon U_j \to \overline{\mathbb{C}}$ and on each intersection $U_i \cap U_j \neq \emptyset$ we have $f_i = \frac{a_{ij} f_j + b_{ij}}{c_{ij} f_j + d_{ij}}$ for some locally constant functions $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ with $a_{ij} d_{ij} - b_{ij} c_{ij} = 1$. If we have $f_i = a_{ij} f_j + b_{ij}$ for locally constant functions $a_{ij} \neq 0$, b_{ij} then $\mathcal{F}$ is *transversely affine* in M ([9]). In few words, a holomorphic foliation $\mathcal{F}$ of codimension one and having singular set $\operatorname{sing}(\mathcal{F})$ of codimension ≥ 2 in a complex manifold M^m, $m \geq 2$ is *transversely projective* if the underlying nonsingular foliation $\mathcal{F}|_{M \setminus \operatorname{sing}(\mathcal{F})}$ is transversely projective on $M \setminus \operatorname{sing}(\mathcal{F})$. Basic references for transversely affine and transversely projective foliations are found in [4].

Remark 1. Assume that the dimension is $m = 2$. Let $q \in \mathrm{sing}(\mathcal{F})$ be an isolated singular point and U a small bidisc such that $\mathrm{sing}(\mathcal{F}) \cap U = \{q\}$. Then $U \setminus \{q\}$ is simply-connected and therefore $\mathcal{F}\big|_{U\setminus\{q\}}$ is given by a holomorphic submersion $f\colon U \setminus \{q\} \to \overline{\mathbb{C}}$ ([9]). By Hartogs' classical extension theorem [5] the map f extends as a meromorphic function $f\colon U \to \overline{\mathbb{C}}$ (possibly with an indeterminacy point at q). Thus, according to our definition, the singularities of a foliation admitting a projective transverse structure are all of type $df = 0$ for some local meromorphic function. For example we can consider $\mathcal{F}$ given in a neighborhood of the origin $0 \in \mathbb{C}^2$ by $kxdy - \ell ydx = 0$ where $k, \ell \in \mathbb{N}$. Then $\mathcal{F}$ is transversely projective in this neighborhood and given by the meromorphic function $f = y^k/x^\ell$. Nevertheless, in this work we will be considering foliations which are transversely projective in the complement of *codimension one invariant divisors*. Such divisors may, a priori, admit other types of singularities. In particular, they can exhibit singularities which do not admit meromorphic first integrals. An example is given by a *hyperbolic singularity* of the form $xdy - \lambda ydx = 0$ where $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The corresponding foliation is transversely projective (indeed, transversely affine) in the complement of the set of separatrices $\{x = 0\} \cup \{y = 0\}$. However, an easy computation with Laurent series shows that the foliation admits no meromorphic first integral in a neighborhood of the origin minus the two coordinate axes.

4.2 Projective transverse structures and differential forms

Let $\mathcal{F}$ be a codimension one holomorphic foliation with singular set $\mathrm{sing}(\mathcal{F})$ of codimension ≥ 2 on a complex manifold N. The existence of a projective transverse structure for $\mathcal{F}$ is equivalent to the existence of suitable triples of differential forms as follows:

Proposition 2 ([9])**.** *Assume that $\mathcal{F}$ is given by an integrable holomorphic one-form Ω on N and suppose that there exists a holomorphic one-form η on N such that* (P1) $d\Omega = \eta \wedge \Omega$. *Then $\mathcal{F}$ is transversely projective on N if and only if there exists a holomorphic one-form ξ on N such that* (P2) $d\eta = \Omega \wedge \xi$ *and* (P3) $d\xi = \xi \wedge \eta$.

This motivates the following definition:

Definition 3. Given holomorphic one-forms (respectively, meromorphic one-forms) Ω, η and ξ on N we shall say that (Ω, η, ξ) is a *holomorphic projective triple* (respectively, a *meromorphic projective triple*) if they satisfy relations (P1), (P2) and (P3) above.

With this notion Proposition 2 says that $\mathcal{F}$ is transversely projective on N if and only if the holomorphic pair $(\Omega,\ \eta)$ may be completed to a holomorphic projective triple. If for a holomorphic projective triple we have $d\eta = 0$ and $\xi = 0$ then the projective transverse structure is indeed an affine transverse structure (cf. [9]). Also according to [9] we may perform modifications in a holomorphic or meromorphic projective triple as follows:

Proposition 3. (i) *Given a meromorphic projective triple (Ω, η, ξ) and meromorphic functions g, h on N we can define a meromorphic projective triple as follows:*

(M1) $\Omega' = g\,\Omega$

(M2) $\eta' = \eta + \frac{dg}{g} + h\,\Omega$

(M3) $\xi' = \frac{1}{g}\left(\xi - dh - h\eta - \frac{h^2}{2}\,\Omega\right)$

(ii) *Two holomorphic projective triples* (Ω, η, ξ) *and* (Ω', η', ξ') *define the same projective transverse structure for a given foliation* $\mathcal{F}$ *if and only if we have* (M1), (M2) *and* (M3) *for some holomorphic functions* g, h *with* g *nonvanishing.*

This last proposition implies that suitable meromorphic projective triples also define projective transverse structures.

Definition 4. A meromorphic projective triple (Ω', η', ξ') is *true* if it can be written locally as in (M1), (M2) and (M3) for some (locally defined) holomorphic projective triple (Ω, η, ξ) and some (locally defined) meromorphic functions g, h.

As an immediate consequence we obtain:

Proposition 4. *A true projective triple* (Ω', η', ξ') *defines a transversely projective foliation* $\mathcal{F}$ *given by* Ω' *on* N.

The uniqueness of a meromorphic projective triple is described by the following lemma from [9]:

Lemma 5. *Let* (Ω, η, ξ) *and* (Ω, η, ξ') *be meromorphic projective triples. Then* $\xi' = \xi + F\,\Omega$ *for some meromorphic function* F *in* N *with* $d\,\Omega = -\frac{1}{2}\,\frac{dF}{F} \wedge \Omega$.

We can rewrite the condition on F as $d(\sqrt{F}\,\Omega) = 0$. This implies that if the projective triples (Ω, η, ξ) and (Ω, η, ξ') are not identical then the foliation defined by Ω is transversely affine outside the codimension one analytical invariant subset $\Lambda := (F = 0) \cup (F = \infty)$. ([9]).

4.3 Solvable groups of local diffeomorphisms

We state a well-known technical result.

Lemma 6. *Let* $G < \mathrm{Diff}(\mathbb{C}, 0)$ *be a solvable subgroup of germs of holomorphic diffeomorphisms fixing the origin* $0 \in \mathbb{C}$.

(i) *If* G *is nonabelian and the group of commutators* $[G, G]$ *is not cyclic then* G *is analytically conjugate to a subgroup of* $\mathbb{H}_k = \left\{z \mapsto \frac{az}{\sqrt[k]{1+bz^k}}\right\}$ *for some* $k \in \mathbb{N}$.

(ii) *If* $f \in G$ *is of the form* $f(z) = e^{2\pi i\lambda}\,z + \dots$ *with* $\lambda \in \mathbb{C}\backslash\mathbb{Q}$ *then* f *is analytically linearizable in a coordinate that also embeds* G *in* $\mathbb{H}_k$.

Proof. (i) is in [3]. Given $f \in G$ as in (ii) then by (i) we can write $f(z) = \frac{e^{2\pi i\lambda}\,z}{\sqrt[k]{1+bz^k}}$ for some $k \in \mathbb{N}$, $b \in \mathbb{C}$. Since $\lambda \in \mathbb{C}\backslash\mathbb{Q}$ the homography $H(z) = \frac{e^{2\pi i\lambda}\,z}{1+bz}$ is conjugate by another homography to its linear part $z \mapsto e^{2\pi i\lambda}\,z$ and therefore f is analytically linearizable. $\square$

4.4 Extension to irreducible singularities

Throughout this section $\mathcal{F}$ will denote a holomorphic foliation induced by a holomorphic one-form Ω defined on a neighborhood of the origin $0 \in \mathbb{C}^2$ and such that $\operatorname{sing}(\mathcal{F}) = \{0\} \in \mathbb{C}^2$. Denote by $\operatorname{sep}(\mathcal{F}, 0)$ the germ of all the separatrices of $\mathcal{F}$ through $0 \in \mathbb{C}^2$. We assume that the origin is an irreducible singularity. This means that in suitable local coordinates (x, y) in a neighborhood U of the origin, we have local normal forms for the restriction $\mathcal{F}\big|_U$ given by (Irr.1) or (Irr.2).

Lemma 7 (nonresonant *linearizable* case)**.** *Suppose that $\Omega = g(xdy - \lambda ydx)$ for some holomorphic nonvanishing function g in U, and $\lambda \in \mathbb{C} \setminus \mathbb{Q}_+$. Let F be a meromorphic function in $U^* = U \setminus \{xy = 0\}$, such that $d\Omega = -\frac{1}{2}\frac{dF}{F} \wedge \Omega$. If $\lambda \notin \mathbb{Q}$ then F extends to U as a meromorphic function, $F = \tilde{c}.(gxy)^{-2}$ for some constant $\tilde{c} \in \mathbb{C}$.*

Proof. First we remark that from equation $d\Omega = -\frac{1}{2}\frac{dF}{F} \wedge \Omega$ we have that the set of poles of F is invariant by Ω. Therefore, since the only separatrices of the form Ω in U are the coordinates axes, we can assume that F is holomorphic in U^*. Fix a complex number $a \in \mathbb{C}$ and introduce the one-form $\eta_0 = \frac{d(xyg)}{xyg} + a(\frac{dy}{y} - \lambda\frac{dx}{x})$ in U. Since $d(\frac{\Omega}{gxy}) = \frac{dy}{y} - \lambda\frac{dx}{x}$ is closed it follows that $d\Omega = \eta_0 \wedge \Omega$. Thus the one-form $\Theta := -\frac{1}{2}\frac{dF}{F} - \eta_0$ is closed meromorphic in U^* and satisfies $\Theta \wedge \Omega = d\Omega - d\Omega = 0$. This implies that $\Theta \wedge (\frac{dy}{y} - \lambda\frac{dx}{x}) = 0$ in U^* and therefore we have $\Theta = h.(\frac{dy}{y} - \lambda\frac{dx}{x})$ for some meromorphic function h in U. Taking exterior derivatives we conclude that $dh \wedge (\frac{dy}{y} - \lambda\frac{dx}{x}) = 0$ in U^* and therefore h is a meromorphic first integral for Ω in U^*. Since $\lambda \notin \mathbb{Q}$ we must have $h = c$, a constant: indeed, write $h = \sum\limits_{i,j\in\mathbb{Z}} h_{ij}x^i y^j$ in Laurent series in a small bidisc around the origin. Then from $dh \wedge (\frac{dy}{y} - \lambda\frac{dx}{x}) = 0$ we obtain $(i + \lambda j)h_{ij} = 0, \forall (i, j) \in \mathbb{Z} \times \mathbb{Z}$ and since $\lambda \notin \mathbb{Q}$ this implies that $h_{ij} = 0, \forall (i, j) \neq 0 \in \mathbb{C}^2$.

This already shows that the one-form Θ always extends as a meromorphic one-form with simple poles to U and therefore the function F extends as a meromorphic function to U. The residue of Θ along the axis $\{y = 0\}$ is given by $\operatorname{Res}_{\{y=0\}} \Theta = -\operatorname{Res}_{\{y=0\}} \frac{1}{2}\frac{dF}{F} - \operatorname{Res}_{\{y=0\}} \eta_0 = -\frac{1}{2}k - (1 + a)$ where $k \in \mathbb{N}$ is the order of $\{y = 0\}$ as a set of zeroes of F or minus the order of $\{y = 0\}$ as a set of poles of F. Thus by a suitable choice of a we can assume that $\operatorname{Res}_{\{y=0\}} \Theta = 0$ and therefore by the expression $\Theta = c(\frac{dy}{y} - \lambda\frac{dx}{x})$ we conclude that, for such a choice of a, we have $0 = \Theta = -\frac{1}{2}\frac{dF}{F} - \eta_0$ and thus $-\frac{1}{2}\frac{dF}{F} = \frac{dx}{x} + \frac{dy}{y} + \frac{dg}{g} + a(\frac{dy}{y} - \lambda\frac{dx}{x})$ and therefore, comparing residues along the axes $\{y = 0\}$ and $\{x = 0\}$ we obtain that $1 + a \in \mathbb{Q}$ and $1 - a\lambda \in \mathbb{Q}$. Since $\lambda \notin \mathbb{Q}$ the only possibility is $a = 0$. This proves that indeed $-\frac{1}{2}\frac{dF}{F} = \frac{dx}{x} + \frac{dy}{y} + \frac{dg}{g}$ in U and integrating this last expression we obtain $F = \tilde{c}(gxy)^{-2}$ for some constant $\tilde{c} \in \mathbb{C}$. This proves the lemma.

□

Remark 2. (i) According to [12], Theorem II.3.1, *a nondegenerate nonresonant singularity $xdy - \lambda ydx + \Omega_2(x, y) = 0$, $\lambda \in \mathbb{C}\setminus\mathbb{Q}_+$, is analytically linearizable if and only if the corresponding foliation $\mathcal{F}$ is transversely projective in $U\setminus\operatorname{sep}(\mathcal{F}, U)$ for some neighborhood U of the singularity.* **(ii)** Let now $\mathcal{F}$ be of resonant type or of saddle-node type. According to [12], Theorem II.4.2, *the foliation admits a meromorphic projective triple near the singularity if and only if on a neighborhood of $0 \in \mathbb{C}^2$, $\mathcal{F}$*

is the pull-back of a Riccati foliation on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ *by a meromorphic map.* The proof of this theorem is based on the study and classification of the Martinet-Ramis cocycles of the singularity expressed in terms of some classifying holonomy map of a separatrix of the singularity. For a resonant singularity any of the two separatrices has a *classifying holonomy* (*i.e.*, the analytical conjugacy class of the singularity germ is determined by the analytical conjugacy class of the holonomy map of the separatrix) and for a saddle-node it is necessary to consider the strong separatrix holonomy map. Thus we conclude that the proof given in [12] works if we only assume the existence of a meromorphic projective triple (Ω', η', ξ') on a neighborhood U_0 of $\Lambda \setminus (0,0)$, where $\Lambda \subset \operatorname{sep}(\mathcal{F}, U)$ is any separatrix in the resonant case, and the strong separatrix if the origin is a saddle-node.

Lemma 8 (general nonresonant case)**.** *Suppose that the origin is a nondegenerate nonresonant singularity of the foliation* $\mathcal{F}$*. Assume that* $\mathcal{F}$ *is transversely projective on* $U \setminus \operatorname{sep}(\mathcal{F}, U)$*. Let* η *be a meromorphic one-form on* U *and* ξ *be a meromorphic one-form on* $U \setminus \operatorname{sep}(\mathcal{F}, U)$ *such that on* $U \setminus \operatorname{sep}(\mathcal{F}, U)$ *the one-forms* Ω, η, ξ *define a true projective triple. Then* ξ *extends as a meromorphic one-form to* U*.*

Proof. By hypothesis the foliation is given in suitable local coordinates around the origin by $xdy - \lambda ydx + \omega_2(x,y) = 0$ where $\lambda \in \mathbb{C} \setminus \mathbb{Q}$, $\omega_2(x,y)$ is a holomorphic one-form of order ≥ 2 at $0 \in \mathbb{C}^2$.

Claim 1. *The singularity is analytically linearizable.*

Indeed, if $\lambda \notin \mathbb{R}_-$ then the singularity is in the Poincaré domain with no resonance and by Poincaré-Linearization Theorem the singularity is analytically linearizable. Assume now that $\lambda \in \mathbb{R}_- \setminus \mathbb{Q}_-$. In this case the singularity is in the Siegel domain and, a priori, it is not clear that the singularity is linearizable. Nevertheless, by hypothesis $\mathcal{F}$ is transversely projective in $U^* = U \setminus \operatorname{sep}(\mathcal{F}, U)$ and by Remark 2 (i) the singularity $0 \in \mathbb{C}^2$ is analytically linearizable. This proves the claim.

Therefore we can suppose that $\Omega\big|_U = g(xdy - \lambda ydx)$ for some holomorphic nonvanishing function g in U. We define $\eta_0 = \frac{dg}{g} + \frac{dx}{x} + \frac{dy}{y}$ in U. Then η_0 is meromorphic and satisfies $d\Omega = \eta_0 \wedge \Omega$ so that $\eta = \eta_0 + h\Omega$ for some meromorphic function h in U. We also take $\xi_0 = 0$ so that $d\eta_0 = 0 = \Omega \wedge \xi_0$ and $d\xi_0 = 0 = \xi_0 \wedge \eta$. The triple (Ω, η_0, ξ_0) is a meromorphic projective triple in U so that according to Proposition 3 we can define a meromorphic projective triple (Ω, η, ξ_1) in U by setting $\xi_1 = \xi_0 - dh - h\eta_0 - \frac{h^2}{2}\Omega = -dh - h\eta_0 - \frac{h^2}{2}\Omega$. Then we have by Lemma 5 that $\xi = \xi_1 + \ell.\Omega$ for some meromorphic function ℓ in U^* such that $d\Omega = -\frac{1}{2}\frac{d\ell}{\ell} \wedge \Omega$.

By Lemma 7 above we have $\ell = \tilde{c}.(gxy)^{-2}$ in U^* and therefore ξ extends to U as $\xi = \xi_1 + \tilde{c}.(gxy)^{-2}$ in U^*. This proves the lemma. $\square$

4.5 Extension from a separatrix of an irreducible singularity

Lemma 9. *Let* Ω *be a holomorphic one-form of type* (Irr.1) *or* (Irr.2) *defined on* U*. Let* $S \subset \operatorname{sep}(\mathcal{F}, U)$ *be a separatrix of* $\mathcal{F}\big|_U$ *which is a strong manifold of* $\mathcal{F}$*, in case* $0 \in \mathbb{C}^2$ *is a saddle-node. Let* F *be a meromorphic function in* U *minus the other separatrix of* $\mathcal{F}$ *in* U *such that*

$d\,\Omega = -\frac{1}{2}\,\frac{dF}{F} \wedge \Omega$. *Then F extends as a meromorphic function to U; indeed we have the following possibilities for Ω and F in suitable coordinates in a neighborhood of the origin:*

(i) $\Omega = g(xdy - \lambda ydx)$ *for some* $\lambda \in \mathbb{C}\setminus\{0\}$ *and some meromorphic function* g. *If* $\{\lambda, \frac{1}{\lambda}\} \cap \mathbb{N} = \emptyset$ *then* $F = \tilde{c}.(gxy)^{-2}$ *for some constant* $\tilde{c}$. *If* $\lambda = -\frac{k}{\ell} \in \mathbb{Q}_-$ *where* $k, \ell \in \mathbb{N}$, $< k, \ell >= 1$ *then* $F = \tilde{c}(gxy)^{-2}.\,\varphi(x^k y^\ell)$ *for some constant* $\tilde{c} \in \mathbb{C}$ *and some meromorphic function* $\varphi(z)$ *in a neighborhood of the origin* $0 \in \mathbb{C}$.

(ii) $\Omega = g.\Omega_{1,\ell} = g(y\,dx + \ell x(1 + \frac{\sqrt{-1}}{2\pi}\,xy^\ell)dy)$ *where* $\ell \in \mathbb{N}$ *and* g *is meromorphic. We have* $F = \tilde{c}.(gx^2y^{\ell+1})^{-2}$ *for some constant* $\tilde{c}$.

(iii) $\Omega = g.\Omega_{(2)} = g(xdy - y^2dx)$ *for some* g *meromorphic. We have* $F = \tilde{c}.(gxy^2)^{-2}$ *for some constant* $\tilde{c}$.

In all cases S *is given by* $\{y = 0\}$ *and the function* F *extends as meromorphic function to a neighborhood of the origin.*

Proof. We define the one-form $\eta = -\frac{1}{2}\,\frac{dF}{F}$. Then η is a closed meromorphic one-form in $U \setminus [\text{sep}(\mathcal{F}, U) \setminus S]$ such that $d\Omega = \eta \wedge \Omega$, moreover the polar set of η is contained in S and has order at most one. If η is holomorphic in $U \setminus [\text{sep}(\mathcal{F}, U) \setminus S]$ then the foliation $\mathcal{F}$ is transversely affine there and therefore the holonomy map h of the leaf $L_0 = S \setminus \{0\}$ is linearizable. Since the origin is irreducible and S is not a central manifold, the conjugacy class of this holonomy map classifies the foliation up to analytic conjugacy ([6],[7]). Thus the singularity is itself linearizable. Assume now that $(\eta)_\infty \neq \emptyset$. In this case we have the residue of η along S given by $\text{Res}_S\,\eta = -\frac{1}{2}\,k$ where k is either the order of S as zero of F or minus the order of S as pole of F. We have two possibilities:

(a) If $-\frac{1}{2}\,k \notin \{2, 3, ...\}$ then according to [9] Lemma 3.1 the holonomy map of the leaf L_0 is linearizable and as above the singularity itself is linearizable.

(b) If $-\frac{1}{2}\,k = t + 1 \geq 2$ for some $t \in \mathbb{N}$ then by [9] Lemma 3.1 there is a conjucacy between the holonomy map of L_0 and a map of the form $h(z) = \frac{\alpha z}{(1+\beta z^t)^{\frac{1}{t}}}$, i.e., this is a finite ramified covering of an homography.

1st. case: Suppose that the singularity is nondegenerate, say $\Omega = xdy - \lambda ydx + ...$ If $\{\lambda, \frac{1}{\lambda}\} \cap \mathbb{N} = \emptyset$ then $a = h'(0) = e^{2\pi i/\lambda} \neq 1$ and by Lemma 6 (ii) the holonomy map h is analytically linearizable. Therefore, as remarked above, in this case the singularity $q_{j_0} \in \text{sing}(\mathcal{F})$ is analytically linearizable. Thus we can assume that $\lambda = -\frac{1}{\ell}$ for some $\ell \in \mathbb{N}$. In this case, either the holonomy is the identity (and therefore linearizable) or there is an analytical conjugacy to the corresponding holonomy of the separatrix $(y = 0)$ for the germ of a singularity $\Omega_{k,\ell} = ky\,dx + \ell x(1 + \frac{\sqrt{-1}}{2\pi}\,x^k y^\ell)dy$ for $k = 1$; such a singularity is called a *nonlinearizable resonant saddle*. Therefore, by [8] and [6] we may assume that $\mathcal{F}|_U$ is of the form $\Omega_{1,\ell} = 0$ in the variables $(x, y) \in U$.

2nd. case: Now we consider the case for which the singularity is a saddle-node. By hypothesis, S is the strong manifold of the saddle-node and therefore its holonomy h is tangent to the identity and thus it is analytically conjugated to $z \to \frac{z}{1+z}$ which is conjugated to the corresponding holonomy

map of the separatrix ($y=0$) for the saddle-node $\Omega_{(2)} = y^2dx - xdy$ so that by [7] the foliation $\mathcal{F}$ is analytically conjugated to $\Omega_{(2)}$ in a neighborhood of the origin.

So far we have proved that the singularity is either analytically linearizable, analytically conjugated to $\Omega_{1,\ell} = 0$ if it is resonant and not analytically linearizable, or analytically conjugated to $\Omega_{(2)} = 0$ if it is a saddle-node. We shall now work with these three models in order to conclude the extension of F to U.
(i) In the linearizable case we can write $S : \{y=0\}$ and $\Omega = g(xdy - \lambda ydx)$ for some holomorphic nonvanishing function g in U. If $\lambda \notin \mathbb{Q}$ then by Lemma 7 F extends meromorphicaly to U. Assume now that $\lambda = -\frac{1}{\ell} \in \mathbb{Q}_-$. Recall that $\eta = -\frac{1}{2}\frac{dF}{F}$ satisfies $d\Omega = \eta \wedge \Omega$ and $d\eta = 0$. If we introduce $\tilde{\eta}_0 = \frac{d(gxy)}{gxy}$ then we have $d\Omega = \tilde{\eta}_0 \wedge \Omega$ and therefore $(\eta - \tilde{\eta}_0) \wedge \Omega = 0$ so that $(\eta - \tilde{\eta}_0) \wedge (\frac{dy}{y} - \lambda\frac{dx}{x}) = 0$ and then $\eta = \tilde{\eta}_0 + H.(\frac{dy}{y} - \lambda\frac{dx}{x})$ for some meromorphic function H in $U_0 := U \setminus \{y=0\}$. Since η and $\tilde{\eta}_0$ are closed we conclude that $d(H.(\frac{dy}{y} - \lambda\frac{dx}{x})) = 0$ in U_0. Write now $H = \sum\limits_{i,j\in\mathbb{Z}} H_{ij}x^iy^j$ in Laurent series in a small bidisc around the origin. We obtain from the last equation that $(i+\lambda j)H_{ij} = 0$, $\forall i,j \in \mathbb{Z}$ (for $\lambda \notin \mathbb{Q}$ this implies, again, that $H = H_{00}$ is constant). Thus we have $\Omega \wedge d(xy^\ell) = 0$ and also $F = \varphi(xy^\ell)$ for some function $\varphi(z) = \sum\limits_{t\in\mathbb{Z}} \varphi_t z^t$ defined in a punctured disc around the origin. Nevertheless, the function F is meromorphic along the axis $\{y=0\}$ and therefore φ extends to the origin $0 \in \mathbb{C}$ as a meromorphic function and thus F extends to a neighborhood of the origin as $F = \varphi(xy^\ell)$.

(ii) In the nondegenerate nonlinearizable case we can write $S : \{y=0\}$ and $\Omega = g\,\Omega_{1,\ell} = g(y\,dx + \ell x(1+\frac{\sqrt{-1}}{2\pi}\,xy^\ell)dy)$ for some holomorphic nonvanishing function g on U. Define $\tilde{\eta}_0 = \frac{d(gx^2y^{\ell+1})}{gx^2y^{\ell+1}}$. As above we conclude that $\eta = \tilde{\eta}_0 + H.(n\frac{dx}{x^2y^\ell} + m\frac{dy}{xy^{\ell+1}} + \frac{m\sqrt{-1}}{2\pi}\frac{dy}{y})$ for some meromorphic function H in U_0 such that $dH \wedge (n\frac{dx}{x^2y^\ell} + \ell\frac{dy}{xy^{\ell+1}} + \frac{\ell\sqrt{-1}}{2\pi}\frac{dy}{y}) = 0$. In other words, H is a meromorphic first integral in U_0 for the foliation $\mathcal{F}$. This implies that H is constant. In order to see this it is enough to use Laurent series as above. Alternatively one can argue as follows. If H is not constant then the holonomy map h of the leaf $L_0 \subset S$ leaves invariant a nonconstant meromorphic map (the restriction of the first integral H to a small transverse disc to S). This implies that h is a map with finite orbits and indeed h is periodic. Nevertheless this is never the case of the holonomy map of the separatrix $\{y=0\}$ of the foliation $\Omega_{1,\ell}$. Thus the only possibility is that H is constant.

(iii) In the saddle-node case we can write $\Omega = g\,\Omega_{(2)} = g(xdy - y^2dx)$ for some holomorphic nonvanishing function g in U. Defining $\tilde{\eta}_0 = \frac{d(gxy^2)}{gxy^2}$ and proceeding as above we conclude that $\eta = \tilde{\eta}_0 + H.(\frac{dy}{y^2} - \frac{dx}{x})$ for some meromorphic function H in $U_0 = U \setminus \{x=0\}$ such that $dH \wedge (\frac{dy}{y^2} - \frac{dx}{x}) = 0$, i.e., H is a meromorphic first integral for the saddle-node in U_0. A similar argumentation as above, either with Laurent series or with holonomy arguments, shows that H must be constant.

We have therefore proved that in all cases $\eta = \tilde{\eta}_0 + H.\omega$ for some meromorphic function H in U and some meromorphic closed one-form ω in U. Moreover, H is constant except in the resonant case. This shows that $\eta = -\frac{1}{2}\frac{dF}{F}$ extends to U as a meromorphic one-form and therefore also F extends to U as a meromorphic function, the lemma is proved. $\square$

Lemma 10. *Fix a separatrix* $\Lambda \subset \operatorname{sep}(\mathcal{F}, U)$ *which is not a central manifold, in case the origin is a saddle-node. Let* η *be a meromorphic one-form in* U *and* ξ *be a meromorphic one-form in* $U \setminus [\operatorname{sep}(\mathcal{F}, U) \setminus \Lambda]$ *such that in* $U \setminus \operatorname{sep}(\mathcal{F}, U)$ *the one-forms* Ω, η, ξ *define a projective triple. Then* ξ *extends as a meromorphic one-form to* U*.*

Proof. The proof is based in the preceding results and in Theorem II.4.2 of [12] (see Remark 2). Let us analyze what occurs case by case:

Nondegenerate singularity: First assume that $\mathcal{F}$ is nondegenerate and nonresonant. By Lemma 8 above the singularity is analytically linearizable and the one-form ξ extends to U as a meromorphic one-form. Now we consider the resonant case, i.e., $\Omega = g(xdy - \lambda ydx + ...)$ with $\lambda = -\frac{n}{m} \in \mathbb{Q}_-$ and that the singularity is not analytically linearizable. As we have seen in Remark 2, $\mathcal{F}$ is the pull-back of a Riccati foliation on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ by some meromorphic map $\sigma \colon U - -- > \overline{\mathbb{C}} \times \overline{\mathbb{C}}$ provided that there is a meromorphic projective triple (Ω', η', ξ') in a neighborhood W of a separatrix $\Lambda \subset \operatorname{sep}(\mathcal{F}, U)$. From our hypothesis such a projective triple is given by the restrictions of Ω and η to $U \setminus [\operatorname{sep}(\mathcal{F}, U) \setminus \Lambda]$ and by the one-form ξ. Thus we conclude that $\mathcal{F}$ is a meromorphic pull-back of a Riccati foliation and in particular there is a one-form ξ' defined in a neighborhood $\tilde{U}$ of the origin such that (Ω, η, ξ') is a projective triple in this neighborhood. This implies that $\xi = \xi' + \ell.\Omega$ in $\tilde{U}$ for some meromorphic function ℓ in $\tilde{U}$ such that $d\Omega = -\frac{1}{2}\frac{d\ell}{\ell}$ in $\tilde{U}$. Now we have two possibilities. Either $\xi = \xi'$ in $\tilde{U}$ or $\ell \not\equiv 0$. In the first case ξ extends to U as a meromorphic one-form, $\xi = \xi'$. In the second case we apply Lemma 9 above in order to conclude that the singularity is analytically normalizable and ℓ extends as a meromorphic function to U. Finally, suppose the singularity is resonant analytically linearizable, that means $\mathcal{F}$ is given in U by $\Omega = g\big(xdy + \frac{n}{m}ydx\big)$ where $n, m \in \mathbb{N}$ and g is a meromorphic function in U. In this case as above we define $\eta_0 = \frac{dg}{g} + \frac{dx}{x} + \frac{dy}{y}$, write $\eta = \eta_0 + h\Omega$ and define $\xi_0 = 0, \quad \xi_1 = \xi_0 - dh - h\eta_0 - \frac{h^2}{2}\Omega = -dh - h\eta_0 - \frac{h^2}{2}\Omega$. Now we have $\xi = \xi_1 + \ell\Omega$ for some meromorphic function ℓ in U^*. In this case we have from $d\ell = -\frac{1}{2}\frac{d\ell}{\ell} \wedge \Omega$ that $\ell(gxy)^2 = [\varphi(x^n y^m)]^2$ for some meromorphic function $\varphi(z)$ defined in a punctured neighborhood of the origin $0 \in \mathbb{C}$. In particular we conclude that since ξ extends to some separatrix $\{x = 0\}$ or $\{y = 0\}$ as a meromorphic one-form then it extends to U as a meromorphic one-form.

Saddle-node case: Finally, we assume that the origin is a saddle-node. We write $\Omega = g[y^{t+1}dx - (x(1+\lambda y^t) + \dots)dy]$ for some holomorphic nonvanishing function g in U. Again by Remark 2 there exists a meromorphic projective triple (Ω', η', ξ') for $\mathcal{F}$ in U which is given by a meromorphic pull-back of a Riccati foliation projective triple. We can assume that $\eta' = \eta$ and therefore $\xi = \xi' + \ell\Omega$ where ℓ is a meromorphic function in U^* such that $d\Omega = -\frac{1}{2}\frac{d\ell}{\ell} \wedge \Omega$. There are two cases: If $\ell \equiv 0$ then ξ extends as ξ' to U. Assume that $\ell \not\equiv 0$. In this case by Lemma 9 the singularity is analytically conjugated to $\Omega_{(t)}$ and the function ℓ extends to U as a meromorphic function. Thus ξ extends as a meromorphic one-form to U. ∎

Lemma 11 (noninvariant divisor)**.** *Let be given a holomorphic foliation* $\mathcal{F}$ *on a complex manifold* M*. Suppose that* $\mathcal{F}$ *is given by a meromorphic integrable one-form* Ω *which admits a meromorphic one-form* η *on* M *such that* $d\Omega = \eta \wedge \Omega$*. If* $\mathcal{F}$ *is transversely projective in* $M \setminus \Lambda$ *for some*

noninvariant irreducible analytic subset $\Lambda \subset M$ of codimension one then $\mathcal{F}$ is transversely projective in M.

Proof. Our argumentation is local, *i.e.*, we consider a small neighborhood U of a generic point $q \in \Lambda$ where $\mathcal{F}$ is transverse to Λ. Thus, since Λ is not invariant by $\mathcal{F}$, performing changes as $\Omega' = g_1\Omega$ and $\eta' = \eta + \frac{dg_1}{g_1}$ we can assume that Ω and η have poles in general position with respect to Λ in U. The existence of a projective transverse structure for $\mathcal{F}$ off Λ then gives a meromorphic one-form ξ in $M \setminus \Lambda$ such (Ω, η, ξ) is a true projective triple in $M \setminus \Lambda$. For U small enough we can assume that for suitable local coordinates $(x, y) = (x_1, ..., x_n, y) \in U$ we have $\Lambda \cap U = \{x_1 = 0\}$ and also

$$\Omega = gdy, \eta = \frac{dg}{g} + hdy$$

for some holomorphic function $g, h \colon U \to \mathbb{C}$ with $1/g$ also holomorphic in U. Then we have

$$\xi = -\frac{1}{g}\big[dh + \frac{h^2}{2}dy\big]$$

where

$$d(\sqrt{\ell}gdy) = 0$$

Thus, $\sqrt{\ell}g = \varphi(y)$ for some meromorphic function $\varphi(y)$ defined for $x_1 \neq 0$ and therefore for $x_1 = 0$. This shows that ξ extends to W as a *holomorphic one-form* and then the projective structure extends to U. This shows that the transverse structure extends to Λ. $\square$

Summarizing the above discussion we obtain the following proposition:

Proposition 5. *Let $\mathcal{F}$ a holomorphic foliation in a neighborhood U of the origin $0 \in \mathbb{C}^2$ with an isolated singularity at the origin. Suppose that $\mathcal{F}$ is transversely projective in $U \setminus \operatorname{sep}(\mathcal{F}, U)$ and let (Ω, η, ξ) be a meromorphic triple in $U \setminus \operatorname{sep}(\mathcal{F}, U)$ with Ω holomorphic in U, η meromorphic in U and ξ meromorphic in $U \setminus \operatorname{sep}(\mathcal{F}, U)$. Then the one-form ξ is infinitesimally extensible with respect to $\mathcal{F}$.*

From Proposition 5 and Theorem 1 we obtain:

Theorem 2. *Let $\mathcal{F}$ a holomorphic foliation in a neighborhood U of the origin $0 \in \mathbb{C}^2$ with an isolated singularity at the origin. Suppose that $\mathcal{F}$ is transversely projective in $U \setminus \operatorname{sep}(\mathcal{F}, U)$ and let (Ω, η, ξ) be a meromorphic triple in $U \setminus \operatorname{sep}(\mathcal{F}, U)$ with Ω holomorphic in U, η meromorphic in U and ξ meromorphic in $U \setminus \operatorname{sep}(\mathcal{F}, U)$. Then the one-form ξ extends as a meromorphic one-form to a neighborhood of the origin provided that it extends to some fundamental domain of $\mathcal{F}$.*

We recall that a germ of a foliation at the origin $0 \in \mathbb{C}^2$ is a *generalized curve* if it exhibits no saddle-node in its resolution by blow-ups ([1]). The generalized curve is *non-resonant* if each connected component of the invariant part of the exceptional divisor contains some singularity of non-resonant type. The inverse image of a fundamental domain of a non-resonant generalized curve contains a fundamental of each singularity arising in its resolution process. Therefore, from Theorem 2 we obtain:

Corollary 1. *Let $\mathcal{F}$ be a germ of a non-resonant generalized curve at the origin $0 \in \mathbb{C}^2$. Suppose that $\mathcal{F}$ is transversely projective in $U \setminus \operatorname{sep}(\mathcal{F}, U)$ and let (Ω, η, ξ) be a meromorphic triple in $U \setminus \operatorname{sep}(\mathcal{F}, U)$ with Ω holomorphic in U, η meromorphic in U and ξ meromorphic in $U \setminus \operatorname{sep}(\mathcal{F}, U)$. Then the one-form ξ extends to U as a meromorphic one-form*

We believe that Theorem 1 might have other applications. For instance, consider two germs of holomorphic vector fields with same set of separatrices and holomorphically equivalent in a neighborhood of the singularity minus the local separatrices. In this situation, Theorem 1 may be an useful tool in the investigation of the existence of a holomorphic equivalence for the germs in terms of their associated projective holonomy groups.

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